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Single-Axis Spacecraft Attitude Maneuvers Using an Optimal Reaction Wheel Power Criterion

S.B. Skaar* and L.G. Kraige†
Virginia Polytechnic Institute and State University, Blacksburg, Va.

Introduction

THE subject of spacecraft attitude control using motor-driven reaction wheels has received the attention of numerous investigators due to the attractiveness of electricity as opposed to expendable thruster fuel. The topics of special-case, nonoptimal, large-angle maneuvers,[1,2] fine-pointing control,[3,4] and optimal strategies for momentum de-saturation[5-7] have been widely studied. But the literature contains few efforts (see Ref. 8 for one example) directed toward the treatment of optimal, large-angle, arbitrary attitude maneuvers. The purpose of this Note is to begin the examination of this problem class by considering the special case of single-axis or slewing maneuvers.

The minimization of the energy consumed by the reaction wheel motors is clearly a worthwhile goal. Although the integral of power over the maneuver time would seem to be the ideal performance index since it represents total mechanical work, it does not yield a unique torque history. Moreover, the total work criterion rewards braking, or negative work, as much as it penalizes positive work. Hence, the index treated herein is the integral of power squared over time.

Minimum Power Formulation for Slewing Maneuvers

Consider the rigid-body configuration of Fig. 1, where $\hat{\ell}$ denotes the principal axis about which the maneuver is to occur, I is the spacecraft mass moment of inertia about $\hat{\ell}$, J is the reaction wheel axial moment of inertial about $\hat{\ell}$, θ and ϕ are wheel and spacecraft inertia angular displacements, and u is the motor torque exerted by the motor on the wheel. The problem to be considered is the determination of $u(t)$ such that the performance index

$$J' = \int_{t_0}^{t_f} P^2 \mathrm{d}t \tag{1}$$

(where P is power) is minimized while boundary conditions on the spacecraft and wheel states at the initial and final times are satisfied. The instantaneous power output of the motor is equal to $u\Omega$, where Ω is the angular velocity of the wheel relative to the spacecraft.

It is convenient to express the integrand of Eq. (1) as a function of the state variables ϕ and θ. To that end, recognize that

$$u = J\ddot{\theta} = -I\ddot{\phi} \tag{2}$$

from which follows

$$u = [IJ/(I+J)](\ddot{\theta}-\ddot{\phi}) \tag{3}$$

The incremental work done by the motor is

$$\mathrm{d}W = u(\mathrm{d}\theta - \mathrm{d}\phi) = u(\mathrm{d}\Delta) \tag{4}$$

where

$$\Delta = \theta - \phi \qquad \dot{\Delta} = \dot{\theta} - \dot{\phi} = \Omega \qquad \ddot{\Delta} = \ddot{\theta} - \ddot{\phi} \tag{5}$$

Substitution of Eq. (3) into Eq. (4) yields

$$\mathrm{d}W = [IJ/(I+J)]\ddot{\Delta}\mathrm{d}\Delta = [IJ/(I+J)]\ddot{\Delta}\dot{\Delta}\mathrm{d}t \tag{6}$$

Now, since $\mathrm{d}W = P\mathrm{d}t$, we are able to write the power P as

$$P = [IJ/(I+J)]\ddot{\Delta}\dot{\Delta} \tag{7}$$

so that Eq. (1) becomes

$$J' = \left[\frac{IJ}{I+J}\right]^2 \int_{t_0}^{t_f} (\ddot{\Delta}\dot{\Delta})^2 \mathrm{d}t \tag{8}$$

Application of the extended Euler-Lagrange equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial F}{\partial \ddot{\Delta}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{\Delta}}\right) + \frac{\partial F}{\partial \Delta} = 0 \tag{9}$$

results in the fourth-order ordinary differential equation

$$\frac{\mathrm{d}^4\Delta}{\mathrm{d}t^4}\left(\frac{\mathrm{d}\Delta}{\mathrm{d}t}\right)^2 + 4\frac{\mathrm{d}^3\Delta}{\mathrm{d}t^3}\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2}\frac{\mathrm{d}\Delta}{\mathrm{d}t} + \left(\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2}\right)^3 = 0 \tag{10}$$

The lengthy analytical integration of Eq. (10) results in

$$\Delta = K_3\left\{-\cos^3\left[\frac{\cos^{-1}(K_1t-K_2)+4\pi}{3}\right] + \frac{12}{5}\cos^5\left[\frac{\cos^{-1}(K_1t-K_2)+4\pi}{3}\right]\right\} - (K_3/4)[K_1t-K_2] + K_4 \tag{11a}$$

and

$$\dot{\Delta} = K_1K_3\cos^2\left[\frac{\cos^{-1}(K_1t-K_2)+4\pi}{3}\right] - \frac{K_1K_3}{4} \tag{11b}$$

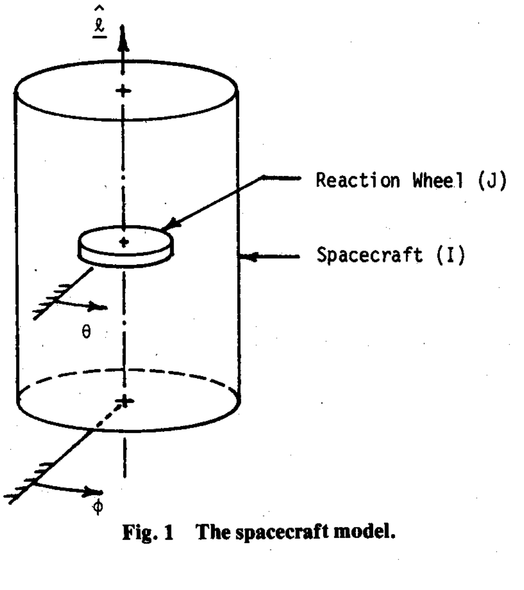

Fig. 1 The spacecraft model.

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*Instructor, Department of Engineering Science and Mechanics. Associate Member AIAA.
†Associate Professor, Department of Engineering Science and Mechanics. Member AIAA.

where K_1-K_4 are constants of integration to be determined in terms of initial and final values of Δ and $\dot{\Delta}$, which in turn must be computed in terms of the known initial and final values of ϕ, $\dot{\phi}$, θ, and $\dot{\theta}$. Using the definition of Δ [Eq. (5)], along with Eqs. (2), the following relationships may be developed:

$$\Delta_0 = \theta_0 - \phi_0 \qquad \dot{\Delta}_0 = \dot{\theta}_0 - \dot{\phi}_0 = \Omega_0$$

$$\Delta_f = \theta_0 - \phi_f + (I/J)(\phi_0 - \phi_f) + [(I/J)\dot{\phi}_0 + \dot{\theta}_0](t_f - t_0)$$

$$\dot{\Delta}_f = \dot{\theta}_0 - \dot{\phi}_f + (I/J)(\dot{\phi}_0 - \dot{\phi}_f) \tag{12}$$

The constant determination process results in

$$\dot{\Delta}_0 = f(P_1,P_2)(P_2^2 - 1/4) \qquad \dot{\Delta}_f = f(P_1,P_2)(P_1^2 - 1/4) \tag{13}$$

where P_1 and P_2 represent a conveniently recombined set of integration constants, and

$$\begin{aligned} f(P_1,P_2) &= (\Delta_f - \Delta_0)/(t_f - t_0) \\ &\quad + [4P_2^3 - 3P_2 + 3P_1 - 4P_1^3][-2P_1^3 + 12/5)P_2^5/ \\ &\quad + (3/4)P_2 - (12/5)P_1^5 - (3/4)P_1 + 2P_1^3] \end{aligned} \tag{14}$$

Equations (13) must be solved numerically for P_1 and P_2. Then K_1-K_4 are given by

$$\begin{aligned} K_1 &= (t_f - t_0)^{-1}[4P_2^3 - 3P_2 + 3P_1 - 4P_1^3] \\ K_2 &= -4P_1^3 + 3P_1 \\ K_3 &= (\Delta_f - \Delta_0)[-2P_2^3 + (12/5)P_2^5 + (3/4)P_2 \\ &\quad - (12/5)P_1^5 - (3/4)P_1 + 2P_1^3]^{-1} \\ K_4 &= K_3[+2P_1^3 - (12/5)P_1^5 - (3/4)P_1] \end{aligned} \tag{15}$$

Finally, from Eq. (3), the optimal control u is given as $[IJ/(I+J)]\ddot{\Delta}$, where $\ddot{\Delta}$ is obtained from differentiation of Eq. (11b).

It should be noted that the four integrations necessary to obtain Eqs. (11) involve the solution of a cubic equation with the attendant branching problems. Equations (11) are valid if boundary conditions are such that P_1^2 and P_2^2 of Eqs. (13) are not greater than one.

A Numerical Example

Consider the following 90-deg rest-to-rest maneuver:

$I = 116$ kg-m^2	$\phi_0 = 0$	$\theta_0 = 0$
$J = 0.05$ kg-m^2	$\dot{\phi}_0 = 0$	$\dot{\theta}_0 = -10$ r/s
$t_0 = 0$	$\phi_f = \pi/2$	
$t_f = 100$ s	$\dot{\phi}_f = 0$	

Note that the final wheel position and angular velocity may not be specified due to the necessity of conserving the system angular momentum.

Figure 2 displays the torque history for the present performance index. Figure 3, added for comparison, shows the torque for the performance index

$$J'' = \int_{t_0}^{t_f} u^2 \mathrm{d}t$$

Additional comparisons are made in Table 1.

For the above and all test cases considered thus far, the power index results in a lower wheel speed and less positive energy expended—both desirable traits. The torque index, however, results in lower instantaneous power and torque requirements.

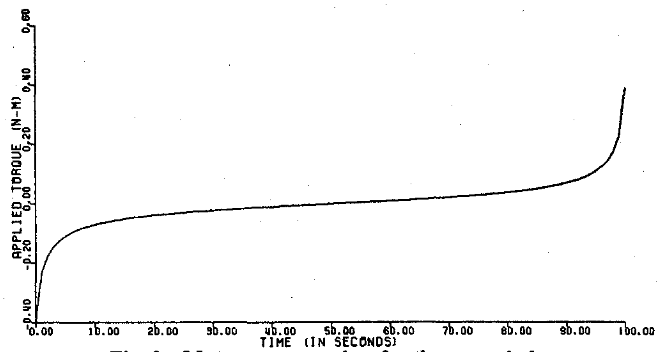

Fig. 2 Motor torque vs time for the power index.

Table 1 Comparison of power and torque optimal criteria

Characteristic	Index $J' = \int_{t_0}^{tf} P^2 \mathrm{d}t$	Index $J'' = \int_{t_0}^{tf} u^2 \mathrm{d}t$
Maximum wheel speed, r/s	57.5	64.7
Total positive energy, J	80.0	102.0
Maximum torque, N-m	0.395	0.109
Maximum power, W	3.95	2.96

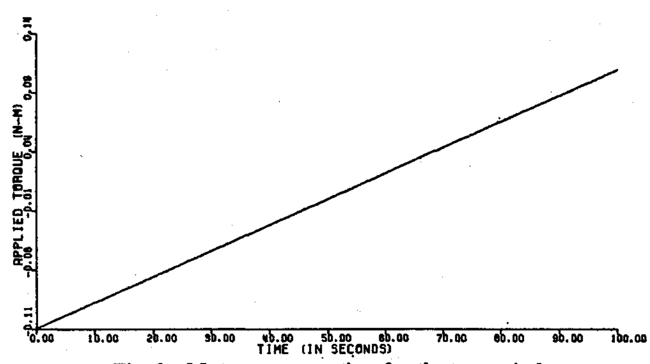

Fig. 3 Motor torque vs time for the torque index.

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